Math 53: Multivariable Calculus **Sections 212, 213**

Worksheet for 2020-09-18

Questions marked with ** are less relevant to the core material and/or more difficult.

Problem 1. A hill is described by the equation

$$z = 50 - 2x^2 + 4x - y^2 - 6y.$$

$f(x,y)$

(a) What are the x and y coordinates of the peak of this hill? (Hint: the tangent plane is horizontal at the peak.)

(b) A person starts at the point $(-3, 0, 20)$ and goes directly towards the peak—meaning their path when viewed from above looks like a straight line. At the point $(-3, 0, 20)$, how steep is their path?

(c) There is a level road around the mountain at elevation $z = 20$ (where the person started walking). What angle is formed between the person's path and the road in $\mathbb{R}^3$? (I didn't try to make the numbers work out nicely so expect inverse trig functions in your final answer.)

(a) Solution 1: look for where $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$:

$$\begin{cases} -4x+4 = 0 \\ -2y-6 = 0 \end{cases} \Longrightarrow x=1, \quad y=-3.$$

Solution 2: Using the fact that this particular surface is an elliptic paraboloid, complete the square to find vertex location:

$$z = -2(x-1)^2 + 2 - (y+3)^2 + 9 + 50$$

So: $x=1$ $y=-3$ @ vertex

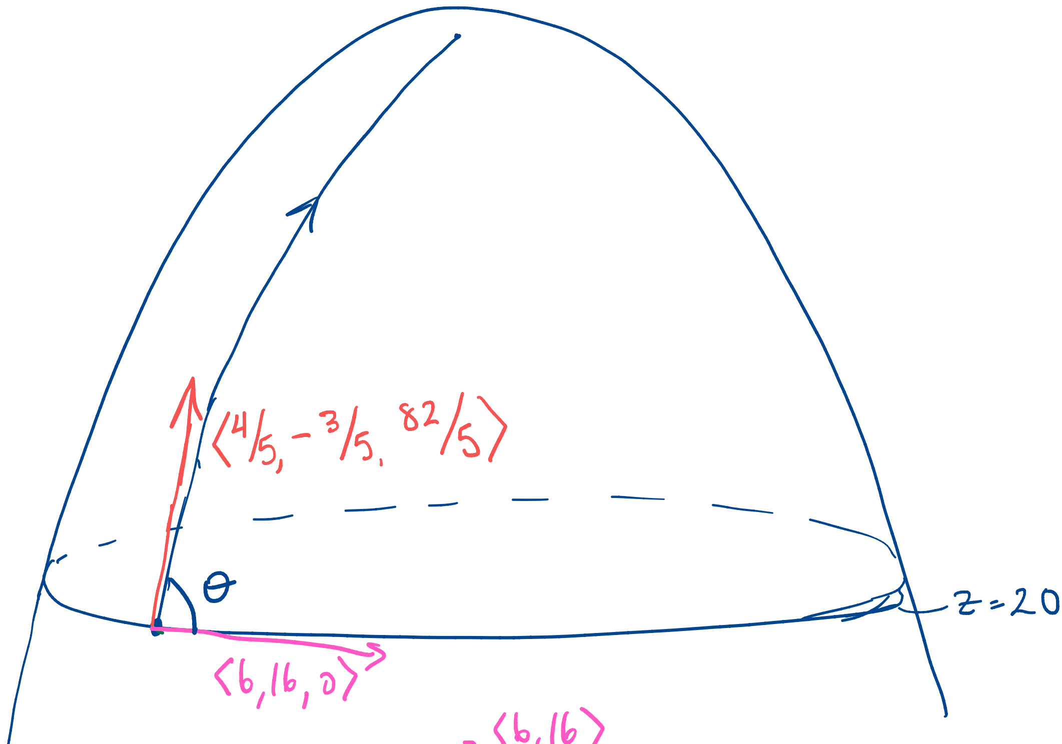
(b) We need to compute $D_{\vec{u}} f(-3,0)$ where $\vec{u}$ is a unit vector in the direction of the peak:

Direction: $\langle 1,-3\rangle - \langle -3,0\rangle = \langle 4,-3\rangle$

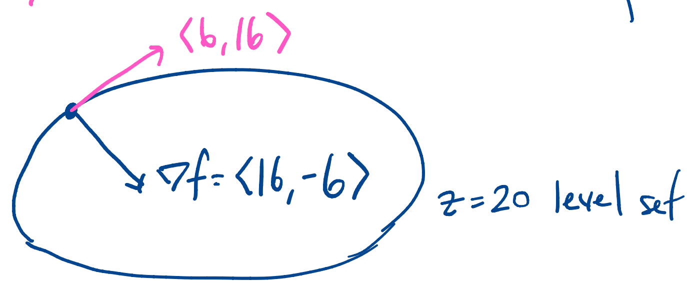
$\vec{u} = \langle 4/5, -3/5\rangle$

$$D_{\vec{u}} f(-3,0) = \langle 16,-6\rangle \cdot \langle 4/5, -3/5\rangle = \boxed{\frac{82}{5}}$$

(c)



View from above:



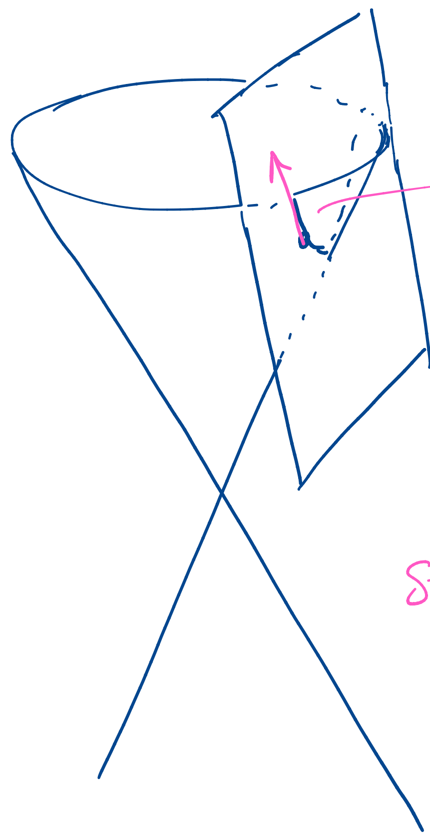
Angle:

$$\cos^{-1}\left(\frac{-24/5}{\frac{\sqrt{6749}}{5}\cdot 2\sqrt{73}}\right) \approx 90.98^\circ$$ so we want

$$180 - 90.98 = \boxed{89.02^\circ}$$ approx.

Problem 2. The cone $x^2 + y^2 = z^2$ intersects the plane $2x + 3y + z = 23$ in a curve C.
Verify that the point $(3, 4, 5)$ lies on C, and find the tangent line to C at that point.
**What kind of curve is C?

Check: $3^2 + 4^2 = 5^2$ and $2 \cdot 3 + 3 \cdot 4 + 5 = 23$.



a tangent vec to C should be tangent to both the cone and the plane.

So it should be orthogonal to the normal vectors of the two surfaces.

Compute a normal to cone:

$F(x,y,z) = x^2 + y^2 - z^2$. $\nabla F(3,4,5) = \langle 6, 8, -10 \rangle$.

Normal to plane: $\langle 2, 3, 1 \rangle$

So a direction vector for tangent line: $\langle 2,3,1 \rangle \times \langle 3,4,-5 \rangle = \langle -19, 13, 1 \rangle$

$$\boxed{\vec{r}(t) = \langle 3,4,5 \rangle + t \langle -19, 13, 1 \rangle}$$

**) Substituting $z = 23 - 2x - 3y$ into first eq:

$$-3x^2 - 12xy + 92x - 8y^2 + 138y - 529 = 0$$

$$(-12)^2 - 4(-3)(-8) = 48 > 0$$

so this is a <u>hyperbola</u>.